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# Mixture Distributions for Modeling Lead Time Demand in Coordinated Supply Chains

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# Panel 16. Defense Supply Chain Modeling Insights

| Thursday, May 15, 2014 | |
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| 11:15 a.m. – 12:45 p.m. | **Chair: Ken Mitchell Jr.,** Director, Research and Analysis, Defense Logistics Agency<br><br>***Mixture Distributions for Modeling Lead Time Demand in Coordinated Supply Chains***<br>Barry Cobb, Virginia Military Institute<br>Alan W. Johnson, Air Force Institute of Technology<br><br>***Maintenance Enterprise Resource Planning: Information Value Among Supply Chain Elements***<br>Rogers Ascef, Naval Postgraduate School<br>Alex Bordetsky, Naval Postgraduate School<br>Geraldo Ferrer, Naval Postgraduate School<br><br>***Multi-Objective Optimization of Fleet-Level Metrics to Determine New System Design Requirements: An Application to Military Air Cargo Fuel Efficiency***<br>Parithi Govindaraju, Purdue University<br>Navindran Davendralingam, Purdue University<br>William Crossley, Purdue University |

# Mixture Distributions for Modeling Lead Time Demand in Coordinated Supply Chains

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## Abstract

This paper introduces a mixture distribution approach to modeling the probability density function for lead time demand (LTD) in problems where a continuous review inventory system is implemented. The method differs from the typical “moment-matching” approach by focusing on building up an accurate, closed-form approximation to the LTD distribution from its components by using mixtures of polynomial functions. First, construction of the lead time distribution is illustrated and the approach is compared to two other possible lead time distributions. This distribution is then utilized to determine optimal order policies in a situation where members of a two-level supply chain coordinate their actions.

## Introduction

This objective of this paper is to describe and implement a mixture distribution method for modeling lead time demand (LTD) in continuous-review inventory problems, then utilize this distribution to jointly determine an optimal order quantity and reorder point. A common approach to finding a probability density function (PDF) for LTD involves modeling lead time (LT) and demand per unit time (DPUT) with standard PDFs. Based on the distributions assigned, a compound probability distribution is determined for demand during LT, or LTD. In some cases, analytical formulas for optimal reorder point, safety stock, or stockout costs are available in terms of the compound distribution’s parameters, while in other situations the values associated with certain percentiles of the compound LTD distribution are estimated to provide these values. While the problem of finding an appropriate LTD distribution has been well studied, researchers in recent years have continued to pursue methods that overcome unrealistic distributional assumptions (Ruiz-Torres & Mahmoodi, 2010; Vernimmen, Dullaert, Willemé, & Witlox, 2008).

There are many situations in which assigning a single, standard PDF as the compound distribution for LTD leads to a poor approximation. Tyworth (1992) observed that this is particularly true when LT is random and follows a non-standard, empirical distribution. This paper illustrates an approach for constructing a mixture distribution for LTD that can incorporate any discrete or continuous LT distribution while approximating the DPUT distribution with a normal PDF. The mixture distribution method for modeling the LTD distribution differs from the typical “moment-matching” approach, as it focuses on constructing the distribution from its components. Use of the mixture distribution technique was demonstrated by Cobb (2013) in a single item continuous-review inventory model for one buyer, and was further described by Cobb and Johnson (2013). Both of these papers illustrated the use of mixtures of truncated exponential (MTE) functions to approximate the DPUT distribution. In this paper, we explore the use of the mixture of polynomials (MOP) approximation as an alternative to the MTE model.

After the mixture distribution approach is described, a two-level supply chain model in which the buyer operates under uncertain demand and utilizes a continuous review inventory system is considered. In this two-echelon supply chain model, credit terms (Chaharsooghi & Heydari, 2010), quantity discounts (Li & Liu, 2006; Chaharsooghi, Heydari, & Kamalabadi, 2011), and rebates (Cobb & Johnson, 2014) have been suggested as incentives that allow the supply chain members to divide the cost savings resulting from coordinating their order quantity and reorder point decisions. In each of these cases, LTD is assumed to be normally distributed. This assumption is not always realistic, particularly when DPUT and LT are each random variables such that LTD has a compound probability distribution (Eppen & Martin, 1988; Lau & Lau, 2003; Lin, 2008). This paper incorporates the mixture model into the two-echelon supply chain problem to show that this model can overcome the requirement that demand for the entire LT period is normally distributed.

The next section discusses the construction of three possible LTD distributions for an example problem in which LT is random. This is followed by a section containing a description of the use of the LTD distribution to find optimal inventory policies in the two-level supply chain. The final section concludes the paper.

## Lead Time Demand Distributions

LTD in a continuous-review inventory system is often assumed to follow a compound probability distribution. Suppose $L$ is a random variable for LT and $D$ represents random DPUT. LTD is a random variable $X$ determined as

$$X = D_1 + D_2 + \cdots + D_i + \cdots + D_L \, . \quad (1)$$

Therefore, $X$ is a sum of random, independent and identically distributed (i.i.d.) instances of demand.

To illustrate the formation of the LTD distribution, we utilize the following example from McClain and Thomas (1985) that has also been used by Eppen and Martin (1988). Demand in each time period is normally distributed with mean $\mu_D = 40$ and variance $\sigma_D^2 = 30$. LT (in periods of one day) may take on the values 7, 12, 14, 15, 16, and 25, and each value has a probability of 1/6.

## Normal Approximation

Because the possible values for LT are dispersed over the range from 7 to 25, the distribution for LTD is multi-modal. As such, there is no one standard PDF that is a good fit. The typical “textbook” approach to modeling the LTD distribution in this case is a normal approximation, and the normal distribution has been used exclusively in the two-stage supply chain model under continuous review assumptions that are presented later in the paper.

The normal approximation to the compound LTD distribution has a mean and variance calculated (Mood, Graybill, & Boes, 1974) as

$$E(X) = E(L) \cdot \mu_D \ \text{ and } \ Var(X) = E(L) \cdot \sigma_D^2 + \mu_D^2 \cdot Var(L). \quad (2)$$

In the example under consideration, $E(L) = 14.83$ days and LT has a variance of $Var(L) = 29.14$. The formulas in (2) are used to determine that $E(X) = 593.33$ and $Var(X) = 47047.2$. If we want to find the reorder point $(R)$ associated with a certain service level (SL), say 95%, we can use the Excel function NORM.INV(0.95,593.33,47047.2^0.5) to find $R = 950$. The SL is the probability that all customer orders are filled in given order cycle.

Eppen and Martin (1988) demonstrated that for this example, implementing $R = 950$ will actually lead to very different SL than 95%. This is because the true distribution of LTD is a mixture of normal distributions. This is discussed in the next section.

## Mixture of Normal Distributions

In this section and for the remainder of the paper, the distribution of LTD is denoted by $f_X$. The distribution of LTD conditional on a specific value $L = l$ for LT is denoted by $f_X|\{L = l\}$. Similarly, the cumulative distribution function (CDF) for LTD is denoted by $F_X$, while the CDF conditional on a specific LT $L = l$ is denoted by $F_X|\{L = l\}$.

In the example problem, if LT is $L = 7$ days, the distribution $f_X|\{L = 7\}$ is a normal PDF with mean 7 × 40 = 280 and variance 7 × 30 = 210. The means and variances of all the conditional LTD distributions can be similarly calculated. The marginal distribution for LTD is the mixture of normal distributions calculated as

$$f_X(x) = \frac{1}{6}\left(f_{X|\{L=7\}}(x) + f_{X|\{L=12\}}(x) + f_{X|\{L=14\}}(x) + f_{X|\{L=15\}}(x) + f_{X|\{L=16\}}(x) + f_{X|\{L=25\}}(x)\right) \quad (3)$$

The mixture of normal distributions for LTD is shown in Figure 1 overlaid on the normal approximation with mean 593.33 and variance 47047.2.

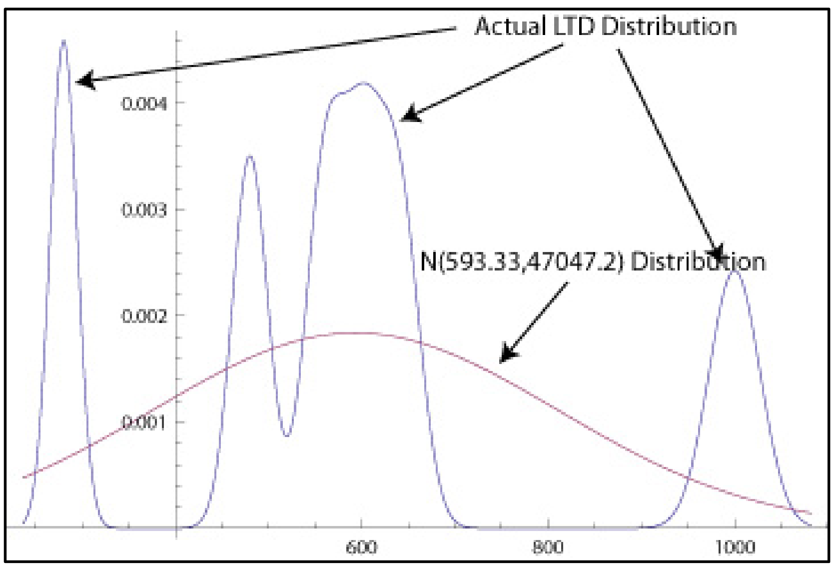


**Figure 1. LTD Distribution and Normal Approximation**

Consider the reorder point $R = 950$. We can find the SL associated with this reorder point by evaluating the conditional CDFs $F_X|\{L = l\}$ at 950 and weighting the results (Eppen & Martin, 1988). This is done as follows:

$$SL(950) = \frac{1}{6}\left(F_{X|\{L=7\}}(950) + F_{X|\{L=12\}}(950) + F_{X|\{L=14\}}(950) + F_{X|\{L=15\}}(950) + F_{X|\{L=16\}}(950) + F_{X|\{L=25\}}(950)\right),$$

$$SL(950) = \frac{1}{6}(1 + 1 + 1 + 1 + 1 + 0.034) = 83.8\%. \quad (4)$$

The conditional values for SL given a certain LT are calculated using the NORM.DIST formula in Excel; for example, the SL given $L = 25$ is NORM.DIST(950,7 × 40,(7 × 30)^0.5,1). Calculation of the reorder point associated with a desired SL cannot be done directly with the exact LTD distribution, but a function such as Goal Seek in Excel can be implemented to find that $R$ = 1014 provides a 95% SL.

## Mixture of Polynomials Approximation

If the functional form of $f_X$ permits closed-form integration, the SL associated with a given reorder point, $R$, can be determined as

$$SL(R) = \int_0^R f_X(x)\, dx\,. \tag{5}$$

Since the functional form of the mixture of normal distributions for the example problem cannot be integrated in this way, built-in Excel functions for the normal CDF were used to calculate the SL. This required weighting the results from the conditional distributions for each possible LT value.

One method for obtaining a closed-form distribution for LTD is the mixture of polynomials (MOP) model (Shenoy & West, 2011). The MOP model can be used to approximate PDFs by piecewise polynomials defined on hypercubes. MOP approximations of standard PDFs, such as the normal distribution, can be developed by using Lagrange interpolating polynomials with Chebyshev points (Shenoy, 2012). This method was used to define a 2-piece, 4th-degree MOP function that approximates the standard normal PDF as

$$g(z) = \begin{cases} 0.398 - 0.038z - 0.322z^2 - 0.148z^3 - 0.020z^4 & \text{if} -3 \le z < 0 \\ 0.398 + 0.038z - 0.322z^2 + 0.148z^3 - 0.020z^4 & \text{if } 0 \le z \le 3. \end{cases} \tag{6}$$

All piecewise functions in this paper are assumed to equal zero in undefined regions. Using this approximation, the PDF for LTD conditional on $L = l$ can be determined as

$$\hat{f}_{X|\{L=l\}}(x) = \frac{1}{\sqrt{l \times \sigma_D^2}}\, g\left(\frac{x - l \times \mu_D}{\sqrt{l \times \sigma_D^2}}\right). \tag{7}$$

The MOP function $\hat{f}_X$ that approximates the PDF $f_X$for LTD is determined as

$$\hat{f}_X(x) = \sum_{i=1}^{k} P(L = l_i) \times \hat{f}_{X|\{L=l\}}(x). \tag{8}$$

The index $i$ has been added to the $k$ possible values for LT. This method can be used when the DPUT distribution is normal, or at least in any situation in which we are willing to approximate the DPUT distribution with a normal distribution. Notice, this would be very different (and more accurate) than approximating the distribution for demand over the entire LT with a normal distribution.

For the example problem, $\hat{f}_X$ is calculated as

$$\hat{f}_X(x) = \frac{1}{6} \times \begin{pmatrix} \hat{f}_{X|\{L=7\}}(x) + \hat{f}_{X|\{L=12\}}(x) + \hat{f}_{X|\{L=14\}}(x) + \hat{f}_{X|\{L=15\}}(x) \\ + \hat{f}_{X|\{L=16\}}(x) + \hat{f}_{X|\{L=25\}}(x) \end{pmatrix} \tag{9}$$

The MOP approximation to the LTD distribution is a relatively compact 15-piece, 4th-degree polynomial defined as

$$\hat{f}_X(x) = \begin{cases} -20.93 + 0.333x - 0.002x^2 + 5.24 \times 10^{-6}x^3 - 5.18 \times 10^{-9}x^4 & \text{if } 236.53 \le x < 280 \\ -45.50 + 0.596x - 0.003x^2 + 6.36 \times 10^{-6}x^3 - 5.18 \times 10^{-9}x^4 & 280.00 \le x < 323.47 \\ \vdots & \vdots \\ -261.34 + 0.996x - 0.001x^2 + 9.03 \times 10^{-7}x^3 - 2.15 \times 10^{-10}x^4 & \text{if } 1000 \le x \le 1082.16. \end{cases} \quad (10)$$

This closed-form function for the LTD distribution is easy to manipulate. It can be easily integrated to find a closed-form function for the CDF of LTD as follows:

$$\hat{F}_X(x) = \begin{cases} 1059.39 - 20.93x + 0.167x^2 - 0.0007x^3 - 1.31 \times 10^{-6}x^4 - 1.036 \times 10^{-9}x^5 & \text{if } 236.53 \le x < 280 \\ 2770.91 - 45.50x + 0.298x^2 - 0.0010x^3 - 1.59 \times 10^{-6}x^4 - 1.036 \times 10^{-9}x^5 & 280.00 \le x < 323.47 \\ 0.1667 & 323.47 \le x < 423.01 \\ 4621.62 - 52.22x + 0.236x^2 - 0.0005x^3 - 5.99 \times 10^{-7}x^4 - 2.692 \times 10^{-10}x^5 & 423.01 \le x < 480 \\ \vdots & \vdots \\ 0.8333 & 705.73 \le x < 917.84 \\ \vdots & \vdots \\ 54808 - 261.34x + 0.498x^2 - 0.0005x^3 + 2.26 \times 10^{-7}x^4 - 4.30 \times 10^{-11}x^5 & \text{if } 1000 \le x \le 1082.16 \\ 1 & x > 1082.16. \end{cases} \quad (11)$$

Using this CDF to find the SL for a reorder point of 950 gives

$$\hat{F}_X(950) = \widehat{SL}(950) = 83.9\%. \quad (12)$$

Evaluating 950 gives $\hat{F}_X$ at each possible reorder point value between $E(X)$ and the first value for $R$ that provides a 95% SL gives $R = 1015$, and this calculation requires 0.05 seconds of computing time.

In summary, the LTD distribution can be modeled using one normal distribution as an approximation over the entire LT period. This method leads to poor results when calculating the SL for a given LT and for finding a reorder point that achieves a targeted SL. The actual distribution for the example problem is a mixture of normal distributions, and Excel formulas and built-in functions can be utilized to find SLs and reorder points, albeit indirectly. The MOP model offers an alternative to constructing a closed-form LTD distribution that can be directly integrated and evaluated to find a CDF for LTD, SLs, and reorder points. As discussed in the remainder of the paper, this distribution can be utilized to find optimal inventory policies in a two-level supply chain under uncertain demand and continuous review assumptions.

## Finding Inventory Policies

Suppose that we want to determine an optimal order quantity and reorder point in a continuous-review inventory system (a "$(Q, R)$" policy). When demand is uncertain, the LTD distribution is an important component of a model used to develop such a policy. Since the MOP technique provides a method for directly representing the LTD distribution in closed form, we will utilize the model developed in the last section. Cobb and Johnson (2013) discussed the use of the MTE model in this context. The MOP method used here is an alternative that has produced a better trade-off between computational complexity and modeling accuracy in our experiments.

In this section, we consider a two-echelon supply chain as depicted in Figure 2. A buyer experiencing random demand places its orders for inventory with the supplier.

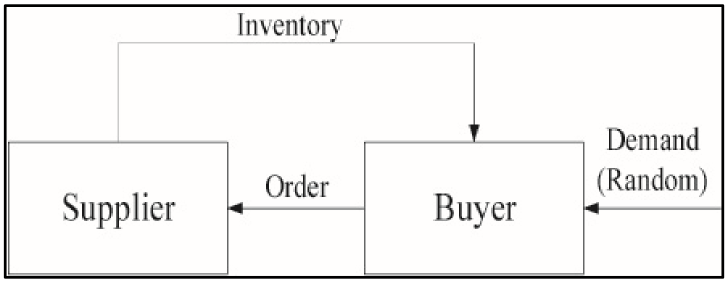


**Figure 2. Two-Echelon Supply Chain**
(Chaharsooghi & Heydari, 2010)

The cost function for the buyer in this problem (Hadley & Whitin, 1963; Johnson & Montgomery, 1974) is

$$TC_b(Q,R,V) = (K_b - V) \cdot \frac{Y}{Q} + \frac{\pi \cdot Y \cdot S_R(R)}{Q} + h_b \cdot (0.5Q + R - E(X)). \quad (13)$$

In this equation, $K$ is the fixed cost per order, $Y$ is the expected annual demand, $h$ is the holding cost per unit per year, and $\pi$ is the stockout cost per unit. The subscript $b$ has been added where necessary to identify assumptions of the buyer. The subscript s similarly represents the seller. The quantity $V$ is a rebate provided by the seller to the buyer on a per-order basis as an incentive for the buyer to adopt policies that benefit both parties (Cobb & Johnson, 2014). As discussed in the introduction, credit options and price discounts have also been considered in this two-level supply chain as coordination incentives (Chaharsooghi & Heydari, 2010; Chaharsooghi et al., 2011; Li & Liu, 2006).

The value $S_R$ in the buyer's cost function is the expected shortage per order cycle, and is calculated for $R$ using the LTD distribution as

$$S_R(R) = \int_R^\infty (x - R) \cdot \hat{f}_X(x)\, dx. \quad (14)$$

Note from Equation 14 that we assume from this point forward that we use the MOP approximation $\hat{f}_X$ to calculate the expected shortage per cycle (i.e., we do not assume we can use the actual PDF $f_X$). The cost function for the supplier in this problem is

$$TC_s(Q,N,V) = \left(\frac{K_s}{N} + V\right) \cdot \frac{Y}{Q} + h_s(N-1)0.5Q. \quad (15)$$

In this two-level supply chain model, the buyer selects an order quantity and reorder point. The supplier receives orders of size $Q$ from the buyer and purchases inventory from its vendors in a quantity that is an integer multiple $N$ of the buyer's order size.

If the buyer selects $Q_d$ and $R_d$ without considering the effect of its selection on the supplier's costs, the supply chain operates in a *decentralized* mode. The supplier simply chooses $N_d$ to minimize its own costs. There is no coordination, so the rebate amount is $V = 0$. Total costs in the supply chain are $TC^d = TC_b(Q_d, R_d, 0)$. + $TC_s(Q_d, N_d, 0)$. Alternatively, if the buyer and supplier compromise on values for $Q_c, R_c$, and $N_c$ that minimize the sum of the cost functions in Equations 13 and 15, this is termed a *centralized* supply chain. There is again no requirement for the supplier to provide a coordination incentive and $V = 0$. Total costs in this mode are denoted by $TC^c = TC_b(Q_c, R_c, 0)$ + $TC_s(Q_c, N_c, 0)$. This type of arrangement is most likely to occur if the buyer and seller are part of the same firm.

If the parties are not centralized but can coordinate their policies, the potential exists to divide cost savings of $TC^{+} = TC^{d} - TC^{c}$. An interval $[V_{min}, V_{max}]$ can be calculated (Cobb & Johnson, 2014) such that any value for the rebate $V$ in the interval reduces the total supply chain costs to centralized levels. The smallest value $V_{min}$ for the rebate the buyer will accept can be found by solving $TC_b(Q_c, R_c, V) = TC_b(Q_d, R_d, 0)$ for $V$. The largest value $V_{max}$ for the rebate the seller will accept can be found by solving $TC_s\ (Q_c, N_c, V) = TC_s\ (Q_d, N_d, 0)\ for\ V$. In this paper, we assume that if the parties agree to coordinate their policies (and implement $Q_c$, $R_c$, and $N_c$), the value of the rebate they select is $\bar{V} = (V_{min} + V_{max}]/2$. Chaharsooghi and Heydari (2010) suggested that division of coordination cost savings could be based on the relative bargaining powers of the two parties.

All of the two-echelon supply chain models referenced previously assume that demand for the entire LT period is normally distributed. For the case where both $Q$ and $R$ are selected to minimize total costs, Charharsooghi and Heydari (2010) derived expressions that state the optimal value for $Q$ (in either the decentralized or centralized mode) as a function of the optimal value for $R$ (and vice versa) and the standard normal CDF. The optimal values can be found by iterating between these two expressions. The supplier selects the integer value for $N$ that minimizes its costs subject to the choices of the buyer.

To implement the MOP mixture distribution approach to find an optimal order quantity/reorder point combination, we first develop a closed-form expression for the expected shortage per cycle in Equation 14 using the previously defined PDF $\hat{f}_X$. This function is an eight-piece, 6th-degree polynomial defined as

$$S_R(R) = \begin{cases} -3.16 \times 10^{-6} + 32362.4R - 138.1R^2 + 0.314R^3 \\ \qquad -0.0004R^4 + 2.75 \times 10^{-7}R^5 - 7.81 \times 10^{-11}R^6 & \text{if } 593.33 \le R < 600 \\ -4.11 \times 10^{-6} + 40257.0R - 164.4R^2 + 0.358R^3 \\ \qquad -0.0004R^4 + 2.87 \times 10^{-7}R^5 - 7.81 \times 10^{-11}R^6 & \text{if } 600 \le R < 621.48 \\ \vdots & \vdots \\ -9.57 \times 10^{-6} + 54807.6R - 130.7R^2 + 0.166R^3 \\ \qquad -0.0001R^4 + 4.52 \times 10^{-8}R^5 - 7.16 \times 10^{-12}R^6 & \text{if } 1000 \le R \le 1082.16. \end{cases} \tag{16}$$

### *Decentralized Solution*

This function for $S_R$ shown above can be substituted into Equation 13 to create a piecewise cost function for the buyer. In this example, we assume $K_b = 50,\ h_b = 5,\ and\ \pi = 6$. Expected annual demand is based on 150 working days and equals $Y = 150 \times \mu_D = 150 \times 40 = 6000$. This cost function is displayed as a function of $Q$ for three values of $R$ in Figure 3. By inspection, we can see that the optimal order quantity is lower for smaller values of $R$. In other words, we can better control costs by simultaneously selecting the order quantity and reorder point.

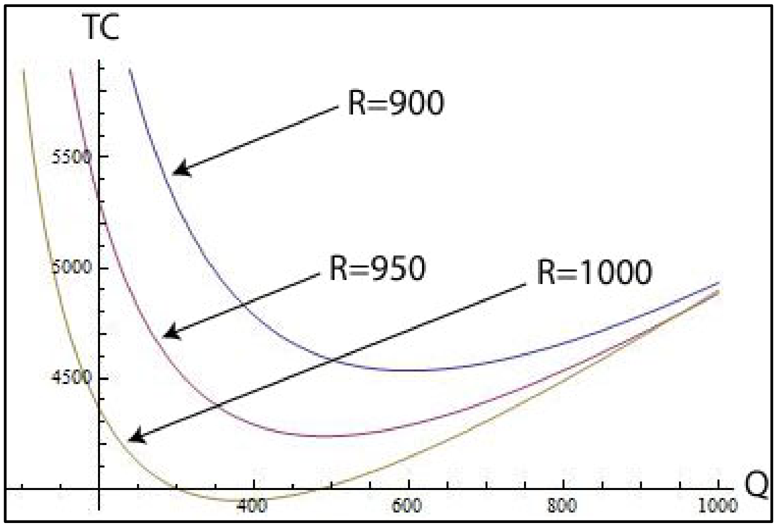


**Figure 3. Buyer's Cost as a Function of Order Quantity for Three Values of the Reorder Point**

### *Decentralized Solution*

Optimization over the cost function developed using the MOP distribution for LTD is fast. Notice that the function for expected shortage per cycle is a MOP. When this expression is inserted in the cost function in Equation 13, the result is a function with polynomial terms and some terms with $Q$ in the denominator. The example here was solved using Mathematica 9.0 by using the ArgMin function. The resulting solutions are $Q_d = 364$ and $R_d = 1014$ with $TC_b(Q_d, R_d, 0) = 3924$. The supplier's best response is to set $N_d = 1$ and incur costs of $TC_s(Q_d, N_d, 0) = 2472$, and total costs in the supply chain are $TC_d = 6396$. The computing time expended is less than one second.

An iterative approach (Hadley & Whitin, 1963) in combination with numerical integration was implemented to find the solutions using the normal approximation to the LTD distribution using the partial solution provided by Chaharsooghi and Heydari (2010). The solutions are $Q_N^d = 447$ and $R_N^d = 925$. If these solutions are inserted in the "actual" cost function (the one developed with the MOP distribution for LTD), the result is $TC_b\left(Q_N^d, R_N^d, 0\right) = 4454$. Using the MOP mixture distribution yields an improvement in costs of 4454–3924 = 530 or 12%.

### *Centralized Solution*

The closed-form function $S_R$ for expected shortage per cycle developed using the MOP distribution for LTD can also be used to derive a cost function for the entire supply chain in the centralized case. This function $TC^C(Q, R, N)$ is used to find the optimal combination $(Q_c, R_c)$ for several possible values of the supplier's decision variable $N$. The value of $N$ producing the lowest total cost once the corresponding optimal values for order quantity and reorder point are selected is deemed the best supplier policy. Typically, solving for the optimal $(Q_c, R_c)$ with $N = 1$ then checking to see if $N = 2$ or $N = 3$ produces a better solution is adequate.

The best order quantity in the centralized model for a given reorder point is higher than the optimal order quantity in the decentralized case. This is illustrated in Figure 4, in which the total costs are graphed as a function of $Q$ for the decentralized and centralized cases assuming a reorder point of $R = 1000$. Visually, the centralized cost function appears to reach a minimum at a larger value of $Q$.

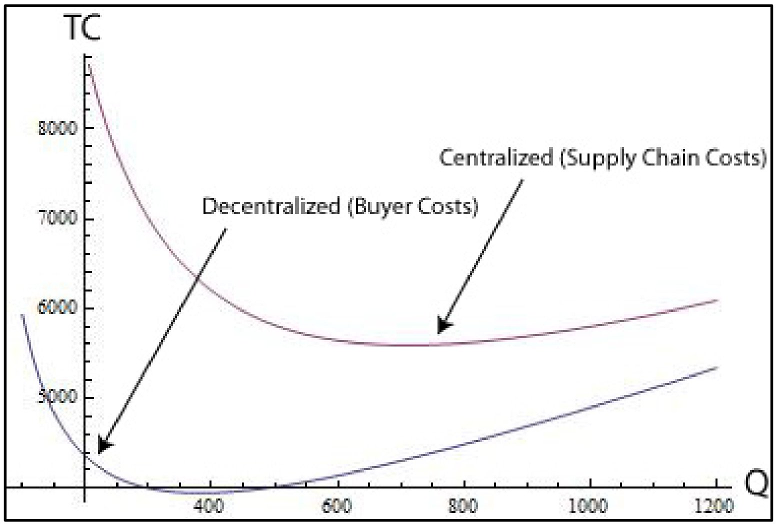


**Figure 4. Decentralized and Centralized Costs as a Function of Order Quantity for a Reorder Point of 1000**

The closed-form centralized cost function can again be easily utilized to find the optimal policy of $(Q_c, R_c, N_c) = (718,993,1)$. The costs for the parties at the optimal solutions are as follows: $TC_b(Q_c, N_c, 0) = 4333$; $TC_s(Q_c, N_c, V) = 1254$; $TC^c = 5587$. The solution again takes around one second of computing time to obtain.

The buyer incurs higher costs by 4333 − 3924 = 409 in the decentralized mode as compared to the centralized mode, where the supplier's costs are reduced by 2472 − 1254 = 1218. Total costs in the supply chain are lower than in the decentralized mode by 6396 − 5587 = 809.

The corresponding centralized solutions found using the normal approximation are $Q_N^c = 802$ and $R_N^c = 857$. If these solutions are inserted in the "actual" cost function for the supply chain (the one developed with the MOP distribution for LTD), the result is $TC_N^c$ = 5891. Using the MOP mixture distribution yields an improvement in costs of 5891 − 5587 = 304 or 5% in the centralized mode.

### *Coordinated Solution*

While the buyer prefers that the supply chain operate in decentralized mode and the supplier wants a centralized solution, both parties can potentially compromise and coordinate to divide the centralized costs savings. The closed-form cost functions developed using the MOP method again provide an approach to determine a supply chain coordination mechanism to make this work.

The buyer will accept a per order rebate as low as $V_{min}$, which can be found by solving $TC_b(718,993, V)$ = 3924, or 4333 − 8.356$v$ = 3924. The solution is $V_{min}$ = 49. The supplier will accept a per-order rebate as high as $V_{max}$, which can be found by solving $TC_s$ (718,993,$V$) = 2472, or 1254 + 8.356$v$ = 2472. The solution is $V_{max}$ = 146.

In this example, at the centralized optimal order quantity, there are $Y/Q_c$ = 6000/718 = 8.356 order cycles per year, so the minimum incentive entails rebates of 8.356 × 49 = 409, and the maximum incentive entails rebates of 8.356 × 146 = 1218. One solution is to implement $\bar{V}$= ($V_{min}$ + $V_{max}$)/2 = 97.5 and require the supplier to provide 815 in rebates to the buyer. This brings the buyer's total costs to 3518, the supplier's total costs to 2069, and supply chain costs to 5587, which is the centralized level.

## Conclusions

This paper has detailed the use of a MOP approximation to the LTD distribution in inventory management problems where a continuous review system is assumed. The use of this approximation was first compared to use of a normal approximation and a mixture of normal distributions approximation. The normal approximation is not very accurate, and the mixture of normal approximation is difficult to manipulate to determine optimal inventory policies. Next, the mixture of polynomials approximation was used to construct a closed-form approximation to the expected shortage per cycle function for the continuous review buyer cost function under uncertain demand. The resulting model was implemented to find optimal inventory policies in a two-level supply chain. The results provide significant cost savings as compared to a solution developed using the normal approximation.

Additional research supported under BAA Number NPS-BAA-12-002 through the Naval Postgraduate School's Acquisition Research Program (Grant N00244-13-1-0014) has extended the use of the MOP approximation to the case where the actual distributions for DPUT and LT are unknown. The approach is the estimate the conditional distributions for LTD using B-spline functions. This methodology will be discussed in the final report presented for that project.

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